The Resonant Response of a Violin G-string and the Excitation of the Wolf-Note

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Summary

Measurements are reported for the mechanical admittance of two G-strings mounted on a violin and excited near the bowing point by a sinusoidal electromagnetic force using a photodetecting device to monitor the resulting string displacement. The violin had a pronounced wolf-note near the main body resonance at $\approx 460$ Hz and the influence of this and other structural resonances on the resonant response of the G-string was investigated from 200 to 550 Hz. Independent measurements of the parameters involved enabled a quantitative comparison to be made with theoretical predictions based on a simple electrical analogue model. In particular, the characteristic response suggested by Schelleng to account for the wolf-note phenomenon has been confirmed. The measurements also clearly demonstrate the existence of two independent modes of transverse vibration of the string, only one of which couples strongly to the lowest body resonance. The different resonant response of these two modes results in the string, in general, being forced to vibrate in a direction different from that of the exciting force.

Das Resonanzverhalten der Violinen-G-Saiten und die Erregung des Wolftons

Zusammenfassung


Réponse d’une corde de sol d’un violon à la résonance et excitation d’une note « hurlée »

Sommaire

On a mesuré l’admittance mécanique de deux cordes de sol montées sur un violon et excitées au moyen d’une force électromagnétique au voisinage du point d’application habituel de l’archet. Les déplacements de la corde étaient enregistrés par un dispositif photo-détecteur. Le violon émettait une note hurlée très marquée vers 460 Hz, fréquence principale de résonance du corps. On a étudié de 200 à 550 Hz l’influence de cette résonance, ainsi que de quelques autres résonances structurelles, sur la réponse de la corde. Par des mesures indépendantes portant sur les divers paramètres en jeu, on a pu vérifier quantitativement certaines prédicitions théoriques dérivées d’un modèle simple fondé sur les analogies électriques. Ainsi on a également retrouvé la courbe de réponse qui caractérise, selon Schelleng, le phénomène de la note hurlée. On a obtenu de même la preuve expérimentale de l’existence de deux modes indépendants de vibration transversale de la corde, l’un d’eux seulement étant couplé à la résonance fondamentale du corps du violon. Ces deux modes possèdent des courbes de réponse à la résonance nettement différentes, d’où il résulte que les vibrations forçées de la corde ne sont en général pas colinéaires à la force excitatrice.

1. Introduction

The bridge on a stringed instrument provides the necessary coupling between the vibrating string and the mechanical modes of vibration of the body which radiate the sound from the instrument. Savart [1] was probably the first to recognise the importance of the bridge in transferring the vibrational energy. A translation of his original paper and many references to subsequent experimental and theoretical work on the acoustics of the violin are contained in the invaluable collection of papers edited by Hutchins [2].
The influence of this coupling on the resonant behaviour of the string was considered theoretically by Raman [3], who also presented a number of most ingenious measurements of the motions of the string and belly of a cello, when the instrument was bowed. Raman was particularly interested in the phenomenon of the wolf-note — a characteristic cyclic variation in the tone of often very fine cellos and violins, when these instruments are bowed at a pitch close to a strong structural resonance of the instrument.

Considerable physical insight into this problem was later provided by Schelleng [4], who analysed the effects of the coupling by treating the violin string as the analogue of an electrical transmission line terminated at the bridge-end by a circuit representing the structural modes of vibration of the instrument. Schelleng showed that, when the coupling between the vibrating string and a particular mechanical resonance was sufficiently strong, the expected single string resonance could be split into a double resonance, symmetrically placed in frequency about the body resonance.

Schelleng suggested that, when the string was bowed under these conditions, both resonances would be excited simultaneously and that their resulting beating would produce the characteristic cyclic variations of the wolf-note. Experiments by Firth and Buchanan [5], on two cellos with pronounced wolf-notes, confirmed several features predicted by Schelleng. In particular, the harmonic analysis of the string vibrations and the sound radiated in the vicinity of the wolf-note was found to be consistent with the predicted splitting of string resonances. However, there have been no measurements made of the resonant response of a string to a sinusoidal excitation in the neighbourhood of the wolf-note, which would allow a very much more direct test of the validity of the Schelleng model. Although Benade [6] has underlined the difficulties involved in such measurements, Hancock [7] has recently succeeded in obtaining some very interesting results, measuring local string velocities by the Doppler-shift of light from a laser, which show that the resonances of a string are considerably more complicated than had hitherto been expected.

In this paper we consider a simple model for string resonances based on the Schelleng analysis. We use this model to interpret measurements of the resonances of a violin G-string excited sinusoidally at the bowing point by a localised electromagnetic force. Measurements of string displacements were obtained using a simple photo-

transistor method that has been described elsewhere [8]. Measurements of the amplitude and phase of string vibrations were compared directly with numerical predictions of the Schelleng model using independent measurements for all the parameters involved.

2. Theory

If we assume a string displacement at a point varying sinusoidally as $x \cos \omega t$, the velocity

$$v_x = j \omega x \cos \omega t.$$

The mechanical impedance $Z$ at a point on the string can be defined as the ratio of the force applied at that point to the resulting velocity of the string,

$$Z = F/v,$$

and the mechanical admittance

$$A = v/F.$$

Schelleng [4] showed that the string could be treated as an electrical transmission line terminated at the bridge end by a transformer coupling to a number of series resonant circuits representing the structural resonances of the instrument. Standard transmission line theory then enables the effective impedance or admittance to be evaluated at any point along the length of the string.

For simplicity we limit our discussion to low frequencies, and assume that only coupling to the lowest strong resonance of the body is important. We will refer to this resonance as the main body resonance. At these frequencies we can consider the bridge as a rigid body and can ignore the frequency dependent transmission characteristics of the bridge [9], which are important at higher frequencies. The problem is then reduced to the simple problem of evaluating the impedance or admittance at a point P on the string using the

![Fig. 1. Simplified equivalent circuit to describe low frequency response of violin string.](image-url)
The equivalent circuit shown in Fig. 1. We may write

\[
\frac{Z}{Z_0} = \frac{\beta}{Q} + j \left[ \beta \delta + \tan \left( \pi \frac{\omega}{\omega_0} \frac{l_1}{l} \right) \right] - \frac{1 - \beta \delta \tan \left( \pi \frac{\omega}{\omega_0} \frac{l_1}{l} \right)}{1 + j \beta \tan \left( \pi \frac{\omega}{\omega_0} \frac{l_1}{l} \right)}
\]

(1)

where \( l_1 \) is the distance between the point of excitation and the bridge; \( l \) is the length of string between the nut, end and the bridge; \( \omega_0 \) is the fundamental resonant frequency of the string assuming a perfect node at each end;

\[
\delta = \left[ \frac{\omega}{\omega_B} - \frac{\omega_B}{\omega} \right],
\]

where \( \omega_B = (MS)^{-1/2} \) is the frequency of the main body resonance, \( M \) and \( S \) being the effective mass and compliance of the mechanical resonance referred to the point of support of the string on the bridge; \( Q = \omega M/R \), where \( R \) represents mechanical and acoustical losses from the body resonance; \( \beta = \omega_B M/Z_0 \), where \( Z_0 = cm/l \) is the mechanical impedance of the string, \( c \) being the velocity of transverse waves on the string and \( m \) the mass of the string length \( l \). For string resonances close to the main body resonance

\[
\beta \approx (n + 1) \pi \frac{M}{m},
\]

where \( n \) is the order of the harmonic excited on the string.

For string resonances not too close to a body resonance or for \( Q^2 \beta < (n + 1) \pi/2 \), the predicted real and imaginary parts of the complex admittance \( \Delta = Z^{-1} \) are similar to the response curves expected for a simple resonance, as illustrated in Fig. 2a, where the position of the string resonance in the absence of any coupling to the body resonance is indicated by an arrow. Despite the weak coupling to the body resonance, the resonant frequency of the string can be appreciably shifted. For the situation illustrated, the string resonance is at a lower frequency than the body resonance and is further depressed by coupling to the body resonance; for a string resonance above the body resonance the resonant frequency would be increased.

This behaviour is easily understood in terms of the motion of the bridge, which is forced to move in-phase with the driving force at frequencies below the structural resonance. The effective node for the coupled mode therefore lies on the far side of the bridge from the vibrating string, increasing the effective resonant length of the string. For string resonances above the structural resonance, the bridge moves in anti-phase with the exciting force, effectively shortening the length and raising the resonant frequency.

For string resonances close to a body resonance, the form of the string resonances can be changed dramatically when the coupling is sufficiently strong, the single resonance splitting into a double resonance. The imaginary component of the admittance can pass through zero not once but three times within the resonance, as illustrated for a
particularly set of parameters in Fig. 2b. For the appearance of double peaks in the real component and multiple zeroes in the imaginary component, \( Q^2/\beta > (n + 1)\pi/2 \), which is essentially the same criterion as that suggested by Schelleng [4] for the appearance of the wolf-note. Figs. 2a and 2b were obtained by numerical evaluation of eq. (1). Recently Clarke [10] has published a rather more detailed mathematical analysis of this equation that was given in Schelleng’s original paper [4].

It is now necessary to consider in a little more detail the way that the bridge transfers energy from the string to the body of the instrument. At low frequencies it is a good approximation to consider the bridge as a mechanical lever, which undergoes a rocking action in its plane about the foot of the bridge closest to the supporting soundpost, as illustrated in Fig. 3. (At higher frequencies it would be necessary to take into account the dynamic characteristics of the bridge itself [9].) This rocking action forces the point of support of the string at \( G \) to move parallel to the \( u \)-direction. Transverse modes of the string vibrations polarised in this direction will therefore couple strongly to the body resonance, and the resonant response of the string should be described by the assumed model. However, transverse modes polarised at right angles to this direction (i.e. vibrating in the \( v \)-direction), will not couple to the body resonance. Such modes will have sharp resonances at unshifted frequencies. The boundary conditions at \( G \) therefore result in two independent solutions for transverse waves on the string, polarised at right angles to each other in the \( u \) and \( v \)-directions. (We assume, of course, a perfect string having degenerate transverse modes in the absence of any coupling through the end supports.)

When the string is excited in any other direction (e.g. the bowing or \( x \)-direction shown in Fig. 3) both modes can be excited simultaneously and in general the transverse motion of the string will not be in the same plane as the exciting force. The response curves for the displacement parallel to a force in an arbitrary direction are therefore superpositions of the resonance curves for the coupled and uncoupled resonances, as illustrated schematically in Fig. 2c. In principle, it is then possible for the imaginary component of the admittance to pass through zero five times within a resonance. Nevertheless, however strong the coupling, there will always be a strong resonance at the natural resonant frequency of the string, the position of which is, to a first approximation, unaffected by coupling to the body resonance.

There has been surprisingly little reference to the various modes of transverse vibration possible on a string, though several authors have shown that the bridge does not in general move in the same direction as the force acting upon it. In particular, Reinicke [9] has studied the response of the bridge over an extended frequency range. In Fig. 2 of his paper, Reinicke presents measurements of the free decay of vibrations of the string for two directions of string polarisation, which can be interpreted as a superposition of the decays of the coupled and uncoupled modes on the string. However, the influence of the uncoupled mode on the formation of stable Raman waves [3] on the bowed string, and the influence of the two modes on the formation of the wolf-note appears not to have been considered in the literature.

3. Experiment

The experiments to be reported were made on an old Italian instrument, which was chosen because its tone in the neighbourhood of the anticipated body resonance at around 460 Hz was rather insecure, when played in high positions on the \( G \)-string. Initially the instrument was strung with a high-quality Pirastro \( G \)-string. This string was later replaced by a heavier, student quality, wire-wound string; when the instrument was then bowed, a pronounced wolf-note could be excited.

To measure the mechanical admittance, resonances were excited electromagnetically by passing a 50 mA current along the string. The string passed between the \( v \)-shaped pole-pieces of a permanent magnet, shown in Fig. 4, giving a localised force at right angles to its field. The magnet could be varied in position between the bridge and the end of the fingerboard and could be rotated in the
plane perpendicular to the wire to give a force in any direction in this plane.

The resulting motion of the string was monitored using a photo-transistor detection system developed by Baker et al. [8]. Separate lamps and photo-transistors were used to monitor the string displacements in the x and y-directions, the shadow of the string moving over the active area of the photo-transistor modulating the current through the device. For convenience, the photo-transistors were permanently mounted on the end of the fingerboard. Since the nut-end of the string is assumed to be a perfect node for the sine-wave excited on the string, it would have been straightforward to have derived velocities at the excitation point from our measurement. However, we have not done so, as, over the relatively small frequency ranges over which measurements are made, the correction factors do not vary appreciably with frequency. We are primarily concerned with the marked frequency dependence in the immediate neighbourhood of the string resonances.

The current through the string was derived from a voltage controlled oscillator, the frequency output of which could be swept linearly through any frequency range of interest by a linear voltage ramp. The oscillator also provided the reference signal for a phase sensitive detector (PSD), which was used to monitor the voltage developed across a resistor in series with the photo-transistor. Over a restricted frequency range the outputs of the PSD at 90° and 180° phase are approximately proportional to the real and imaginary components of the mechanical admittance (strictly speaking these outputs are proportional to $A/\omega$). The output of the PSD was plotted on the XY-recorder as a function of the voltage applied to the oscillator, which was simply proportional to the change in excitation frequency. Frequencies were measured to an accuracy of 0.1 Hz.

Initially measurements were made with an adjustable stop to vary the length and resonant frequency of the string. However, the resulting curves varied with the type of stop used and it was difficult to obtain reproducible results. No stop was as good as the violin’s own nut in ensuring a perfect node for all polarisations of transverse waves on the string. Fret-like stops were particularly bad, and the observed response curves could be changed dramatically simply by adjusting the pressure on the string behind the stop. This may play an important role in the technique of playing fretted instruments.

To avoid such complications from imperfect stops, we confined our measurements to resonances of the total string length between nut and bridge, varying the resonant frequency by adjusting the tension in the string and using higher harmonics to extend the frequency range.

4. Measurements

In Fig. 5 we show a typical set of measurements of the string displacement parallel to a force in the bowing or x-direction. Such curves are consistent with there being two independent resonant modes of the string, as expected from our previous discussion. Further support for such an interpretation comes from measurements such as those shown in Fig. 6, where the x-displacement has been measured for several angles of the field in the xy-plane. When the exciting force was at

Fig. 5. A typical set of resonance curves for the string displacement at 90° and 180° phase with respect to exciting force. The 90° signal is in phase with the component of velocity in phase with the exciting force.
about 40° to the bowing direction (between positions 2 and 3) a single rather broad resonance could be excited. We identify this resonance as the coupled string resonance polarised in the rocking direction (the u-direction in Fig. 3). For the exciting force at approximately 90° to this (position 7), a much sharper resonance at a higher frequency can be excited, which we identify as the uncoupled resonance. For other directions of the exciting field, the response can be interpreted as a linear superposition of these two resonances. Measurements of the amplitude and phase of displacements in the x and y-directions confirmed that the observed modes were linearly polarised.

By careful adjustment of the field direction it was possible to study the resonant response of the coupled modes alone over an extended range of resonant frequencies. A typical set of exploratory measurements made in this was are shown in Fig. 7. The resonances were scanned relatively fast (about 30 s for each resonance) leading to some ringing for the sharper resonances.

A very pronounced minimum in the heights of the resonances occurs at around 460 Hz, at approximately the frequency that the main body resonance is expected. Smaller minima are also observed at around 290 Hz, almost certainly from the air resonance, and at 310 and 390 Hz, which are probably associated with weak structural resonances. In this paper we confine our attention to string resonances in the vicinity of the main body resonance. In this region, the resonance curves could be changed dramatically by adding small weights to the bridge or belly of the instrument, thereby shifting the frequency of the main body resonances [11].

A sequence of measurements of \( x(F_\alpha) \) in the immediate neighbourhood of the assumed body resonance at \( \approx 460 \) Hz is shown in Fig. 8. These may readily be interpreted in terms of a superposition of coupled and uncoupled resonant modes. The relative position of the coupled and uncoupled string resonances changes depending on whether the string resonance lies above or below the body resonance, as discussed in section 2. When the frequency of the string resonance is close to that of the main body resonance the coupled resonance is appreciably broadened. Numerical evaluations of eq. (1) show that the detailed form of the resonance curves in this region is markedly dependent on the value of \( Q^2/\beta \).
To make a critical comparison with the predictions of eq. (1), it is necessary to obtain independent measurements of the various parameters involved. These were obtained from an analysis of the vibration spectrum of body resonances using a HP 5451 B Fourier Synthesizer. Transient vibrations of the belly of the instrument, following a sharp tap to the bridge, were monitored by a light bimorph strip taped to the edge of the instrument, where it was not expected to influence the vibrational modes significantly. The voltage developed across the bimorph strip was digitised and the vibrational spectrum computed automatically within the Fourier Synthesiser. Fig. 9 gives the resulting spectrum in the immediate vicinity of the main body resonance. To a good approximation the Fourier spectrum can be described by a single mechanical resonance at 456 Hz with a value for $Q \approx 19$. There are clearly also small contributions from much weaker resonances at $\approx 420$ Hz and $\approx 485$ Hz, which, in contrast to the main resonance, were not significantly affected when an additional 1 g mass was placed on the bridge alongside the G-string. From the depression in frequency of the main resonance with added mass, a value for the effective mass of the body resonance (measured at the point of support of the string on the bridge) was obtained, giving $M = (10.6 \pm 0.5)$ g (the difference in the height of the two maxima is not significant, since the height is simply proportional to the strength of the tap given to the bridge). The first harmonic of the G-string resonance was purposely tuned to $\approx 400$ Hz, where it does not significantly affect the resonance curves in the frequency range of interest. It was fortunate that the main body resonance was rather well described by a single resonance curve, as in general this might not be the case. Comparison with our simple model would then have been made much more difficult.

The string resonance shown in Fig. 8 were obtained with the violin strung with a high-quality, metal covered, Pirastro, G-string with $m = 0.77$ g. The structural resonance curves were obtained after the violin had been re-strung with a heavier, student quality, wire-wound string with $m = 1.38$ g. Inserting the appropriate values into eq. (1) we can derive a series of resonance curves very similar to those shown in Fig. 8. The agreement can be made nearly perfect, if we take the slightly different value for $\omega_B = 465$ Hz and a value for $Q = 17$; curves for these parameters are shown in Fig. 10. The small difference between these parameters and
those obtained from the structural resonances is probably not significant, since \( \omega_B \) will undoubtedly be influenced slightly by any variation of tension in the non-resonant lengths of strings on the instrument. The value of \( Q \) may well have been affected by any slight movement of the bridge when the G-string was changed or by frequency dependent contributions from any underlying weak structural resonances.

The effect of increasing the coupling between the vibrating string and the body resonance could be studied by using a lower string harmonic or a heavier string. In Fig. 11 we show a sequence of relatively fast scans of string resonances in the neighbourhood of the body resonance for the first harmonic \( (n=1) \) of the heavier string referred to previously. Close to the body resonance, the amplitude of the coupled resonance becomes rather small and very broad with a suggestion of the predicted double resonance in the lowest set of traces. In Fig. 12 we show some measurements at a higher sensitivity, using a longer scanning period, showing a sharp uncoupled resonance straddled by the predicted double resonance of the coupled mode. To investigate the frequency dependence of the coupled resonance in the region between the two maxima, it is necessary to suppress the uncoupled mode by careful adjustment of the
scaling factor for the real and imaginary components of the admittance. The excellent agreement between measurement and theory gives us considerable confidence in the validity of the simple analogue for describing string resonances at relatively low frequencies.

When the violin strung with the heavier G-string was bowed, a pronounced wolf-note could be excited not only at \( \approx 460 \) Hz but also at the sub-harmonic at \( \approx 230 \) Hz. A typical trace of the string displacement for the 460 Hz wolf-note is shown in Fig. 15. It was not possible to establish a perfectly periodic wolf-note on this instrument and no attempt has therefore been made to analyse the observed traces in any detail, other than to note that the typical period between maximum intensity (\( \approx 46 \) ms) is larger than predicted from Schelleng’s analysis [4], the predicted period being simply the inverse of the separation of the outer two zero-crossing frequencies in the imaginary

Fig. 14. Numerical evaluations of eq. (1) superimposed on experimental traces. \( Q = 19, \frac{M}{m} = 7.6, n = 1, \frac{l_1}{l} = 0.15 \).

field direction. One such set of measurements is shown in Fig. 13, which are clearly of the form first proposed by Schelleng [4] illustrated in Fig. 2b. In Fig. 14 we have superimposed on these measurements numerical evaluations from eq. (1) using the parameters derived from measurements of the Fourier spectrum of the structural vibrations. The only adjustable parameter used is the common

\[ (\omega - \omega_0)/\omega_0 \]

Fig. 16. Computed resonance curves showing the influence of varying the point of excitation. \( Q = 19, \frac{M}{m} = 7.6, n = 1 \).

- - - - - \( l_1/l = 0.1 \),

- - - - - \( l_1/l = 0.15 \),

- - - \( l_1/l = 0.2 \).

Fig. 15. String displacements for the 460 Hz wolf-note.
part of the admittance curve for the coupled resonance. For the symmetrical resonance curves shown in Fig. 13 this is about 30 Hz giving an expected period of 33 ms.

However, both our measurements and the numerical evaluations of eq. (1) shown in Fig. 16 show that the position and separation of the zero-crossing frequencies in the imaginary part of the admittance depend critically on the point of excitation of the string. This may help to explain why it was so difficult to excite a stable and reproducible wolf-note on this particular instrument, since the exact position at which the zero-crossing frequencies are equally spaced (assumed to be the condition for excitation of the stable wolf-note) is very sensitive to the position of the point of excitation on the string and moreover varies from one note to the next.

5. Discussion

In view of the success of the simple analogue in describing interactions between the vibrating string and the main body resonance, it would be interesting to extend the measurements to investigate coupling to the air resonance and to the higher structural resonances of this violin as well as to other violins and cellos.

We have shown that, in considering the transverse vibrational modes on a string, it is important to take into account the rocking motion of the bridge, which results in the excitation of two independent modes of transverse vibration at low frequencies. The uncoupled resonances occur at the natural harmonics of the fundamental string resonance; they may therefore play an important role at all frequencies in establishing a stable Raman wave [3] with a saw-tooth wave-form, the period of which is not significantly affected (other than in the immediate vicinity of the wolf-note) by coupling to structural resonances.

In most theoretical treatments of the bowed string (see McIntyre and Woodhouse [13] for a discussion of recent work and for references to earlier studies) it is assumed that the plane of motion of the strings and of the bridge, at the point over which the string passes, is parallel to the exciting force. It is then only necessary to consider the motion of the string in the bowing direction. However, our measurements demonstrate that this assumption will not in general be justified.

Hancock [14] was the first to point out that the rocking action of the bridge will inevitably complicate any discussion of the bowed string. For example, when the discontinuity in velocity excited by the bowing action reaches the bridge, the impulsive force will excite a large number of structural resonances, each with its characteristic frequency. Each structural resonance will cause the bridge to move in a certain direction, so that waves will be generated at the point of support of the string with frequencies and polarisations determined by the structural resonances of the instrument. The components of these waves perpendicular to the bowing direction will be perfectly reflected when they reach the bow, since at that point the string is forced to move in the bowing direction. These reflected components affect the perpendicular force between bow and string, thus modifying the frictional force and string velocity in the bowing direction. If the distance between the bridge and the bowing point is appropriate, a resonance of a particular mode could build up; this might, for example, explain the structure at eight times the fundamental frequency observed in the traces shown in Fig. 15. These reflected transverse waves may be just as important as torsional waves in producing the additional high frequency structure often observed on stable Raman waves [12]. It would clearly be interesting to consider the influence of these reflected waves on the recent important theoretical work on the bowed string by McIntyre and Woodhouse [13] and Schumacher [15].

Finally, we note that the long sound of a plucked string arises from the high-Q uncoupled resonance of the string. Although this mode does not couple to the main body resonance, it couples weakly to other structural resonances of the body involving rocking motions of the bridge about slightly different positions. Energy is therefore transferred weakly from the uncoupled resonance to these other structural modes, which then radiate sound over a relatively long period.

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Note in proof: A rather similar effect has recently been observed for strings on a piano by Weinreich 16.
References
