The Theory of String Resonances on Musical Instruments

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Summary
A model is developed for the resonant response of a lightly damped string coupled at either end to the structural and other resonances of a musical instrument. Analytic expressions are derived for the admittance at the point of sinusoidal excitation. The perturbation in resonant frequency and the damping of the string resonances are related to properties of the structural resonances that ultimately determine the tonal quality of an instrument. In discussing strongly coupled string resonances, we consider the normal modes of the vibrating system and this is extended to include interactions between sympathetically tuned strings. We show that the coupling at the bridge removes the degeneracy of the transverse vibrations of the string. Our discussion is applicable to string resonances on any musical instrument and, where relevant, results are considered from both scientific and musical viewpoints.

Die Theorie von Saitenresonanzen bei Musikinstrumenten

Zusammenfassung
Es wurde ein Modell für das Resonanzverhalten einer schwach gedämpften Saite entwickelt, die an beiden Enden mit der Hauptresonanz und anderen Resonanzen eines Musikinstrumentes gekoppelt ist. Analytische Ausdrücke für die Admittanz an dem Punkt mit sinusförmiger Auslenkung werden hergeleitet. Die Störung der Resonanzfrequenz und die Dämpfung der Saitenresonanzen stehen in Beziehung mit Eigenschaften der Hauptresonanz, welche letztlich die klangliche Qualität eines Instruments mitbestimmt. Bei der Diskussion stark gekoppelter Saitenresonanzen betrachten wir die Normalmoden des Schwingungssystems, was auf die Wechselwirkungen zwischen mitschwingenden Saiten ausgedehnt wird. Wir zeigen, daß die Kopplung am Steg die Entartung der transversalen Schwingungen der Saite aufhebt. Unsere Diskussion ist anwendbar auf Saitenresonanzen jedes Musikinstrumentes. Soweit relevant, werden die Ergebnisse sowohl vom wissenschaftlichen als auch vom musikalischen Standpunkt aus betrachtet.

La théorie des résonances des cordes sur un instrument de musique

Sommaire
On a développé un modèle fournissant la réponse résonante d’une corde faiblement amortie et couplée en chacune de ses extrémités aux résonances, structurelles ou autres, d’un instrument de musique. On obtient des expressions analytiques de l’admittance au point où est appliquée une excitation sinusoidale. La perturbation de la fréquence de résonance et l’amortissement des résonances de la corde sont reliés aux propriétés des résonances structurelles qui déterminent en dernier ressort la qualité tonale d’un instrument. Pour discuter les résonances à couplages serrés de la corde, on prend en considération les modes normaux du système vibrant. Cette approche a été étendue de manière à inclure les interactions entre cordes accordées de façon à vibrer en sympathie. On montre également que le couplage au niveau du chevalet élimine la dégénérescence des vibrations transversales de la corde. La présente discussion est applicable aux résonances des cordes de tout instrument et ces résultats sont significatifs aussi bien du point de vue musical que du point de vue scientifique.

1. Introduction
The study of the resonances of transverse waves on a stretched string constitutes one of the oldest of all experiments in physics and continues to provide the most common introduction to the theory of vibrations and waves in extended systems, most notably in the account given by Rayleigh [1]. Such resonances also have many applications, particularly in the great diversity of stringed musical instruments, where sounds are produced by plucking, striking or bowing a stretched string (see Benade [2] for a comprehensive account of the physics of such instruments).

In practice, the resonant response of a sinusoidally excited string is strongly influenced by the mechanical properties of the end-supports [1]. On most stringed instruments, the string is supported at one end by a bridge, which, in addition to providing an approximate nodal position for transverse string resonances, also couples energy from the vibrating string to the structural resonances of the instrument. These resonances are ultimately
responsible for the intensity and quality of sound produced, since the string alone radiates a negligible amount of sound.

The influence of such coupling on the transverse wave resonances of a string has been studied by several authors, including Raman [3] and Schelling [4], who were particularly interested in the resonant response of a string when strongly coupled to the main body resonance of instruments of the violin family. When such coupling is sufficiently strong, the string resonances are so strongly perturbed that it no longer becomes possible to excite stable vibrations of the string by bowing, resulting in the well-known wolf-note phenomenon that mars the tone of many an otherwise fine instrument, in the frequency range near the main body resonance (see Benade [2], chapter 25, for an interesting discussion of this phenomenon).

In view of the fundamental importance of string resonances in both physics and music, it is rather surprising that not until very recently has it proved possible to determine the resonant response of strings mounted on musical instruments, or even on the sonometers to be found in almost any student laboratory, with any accuracy. The anticipated difficulty of making such measurements has been discussed by Benade [2], page 510.

However, relatively recently two rather different optical techniques have been developed, which enable measurements of both the amplitude and phase of the string motion near resonance to be determined for the first time. In the first such method, Hancock [5] uses the Doppler shift of light from a laser reflected from the moving string to determine its velocity near resonance. Hancock’s preliminary measurements immediately showed that the transverse vibrations of the string were rather more complicated than anticipated from the usual textbook discussions of the problem. More recently, Baker, Thair and Gough [6] have developed a rather simple photo-detection technique, which allows the response to be determined with extremely high resolution and sensitivity throughout the resonance region.

Using the latter technique, we have already reported [7] a number of measurements of string resonances for a string mounted on a violin with a pronounced wolf-note, arising from a main body resonance that is over-strongly coupled to the string. These measurements confirmed the expected double string resonance in the neighbourhood of the main body resonance, which we associate with the split resonances of the normal modes of the coupled oscillator system formed by the string and main body resonance [4], [8]. From the observed splitting and damping of the resonances, we were able to derive reliable values for the resonant frequency, \( \omega_B \), the quality factor, \( Q_B \), and the effective mass at the point of bridge support, \( M_B \), for the main body resonance. In deriving these values, we compared our measurements of the response near resonance with computed values obtained from a transmission line model for the string first introduced by Kock [9] and applied to the physics of the violin by Schelling [4].

In the above investigation, we were also able to show that the directional nature of the coupling at the bridge removes the degeneracy of transverse string vibrations, resulting in two independent modes of transverse string vibration with different resonant frequencies and polarised in orthogonal directions. The existence of two such modes explains some of the puzzling features reported by Hancock in both recent [5] and earlier measurements [10].

Preliminary measurements of string resonances over a wide range of frequencies suggested that it might be possible to obtain information about other structural resonances coupled less strongly to the string than the main body resonance, by measuring the increased damping of string resonances in the vicinity of the weakly coupled resonances. This has been confirmed by subsequent measurements and will be reported in a separate paper [11] giving experimental illustrations of many aspects of the resonances of transverse waves discussed here.

In this paper we provide a general framework for the interpretation of such measurements and show how the results obtained can be used to derive quantitative information about any structural resonance that is sufficiently strongly coupled to the vibrating string to significantly influence the sound produced by an instrument. Our discussion is not restricted to resonances of strings on instruments of the violin family, but is applicable to all types of stringed instrument, including such varied examples as the guitar, banjo, zither, harp and piano.

The vibrating string is considered as a simple mechanical oscillator, which to a good approximation vibrates in a single mode when excited at frequencies close to one of its resonant frequencies. A comparison with exact results shows that such an approximation is usually justified for lightly damped string resonances on a musical instrument; provided measurements are not made too close to a vibrational mode of the principal resonance excited.

Such a model enables us to obtain simple analytic expressions for the response near resonance, which are easier to interpret than numerical results obtained by Hancock [10] and Clarke [12] using the
transmission line model. Furthermore, our approach allows us to consider the excitation of a string by a distributed exciting force and to discuss the consequences of the directional coupling at the bridge. In contrast, the transmission line model generally assumes a point driving force and a string motion parallel to the exciting force, neither of which is justified in practice.

To introduce our model we first consider the resonances of a lossy but perfectly flexible string between rigid end-supports. We then examine the influence of a yielding end-support which allows coupling to the structural resonances of the instrument on which the string is mounted. We show that, when this coupling is relatively weak, the induced motion causes a small change in the effective length of the string and some additional damping. The resulting change in resonant frequency and additional damping can be related to \( \omega_B \), \( Q_B \) and \( M_B \) of the coupled resonance. We indicate how measurements of string resonances near a weakly coupled structural resonance can be used to derive these parameters.

For sufficiently strong coupling, the perturbation in string length and the additional damping can no longer be assumed to be constant over the width of the string resonance. It then becomes necessary to describe the string resonances in terms of the normal modes of the coupled string and structural resonances. For string resonances close to a strongly coupled structural resonance, the string exhibits the double resonance referred to earlier, with a splitting and damping that can again be simply related to the values of \( \omega_B \), \( Q_B \) and \( M_B \) for the coupled resonance.

As a direct consequence of the directional nature of the coupling between the string and structural vibrations at the bridge, the degeneracy of the transverse string vibrations is removed. We consider the possible modes of string vibration for a string coupled at either end to a number of structural resonances and indicate how measurements of the polarisation of the modes excited on a string can be expected to provide information about the coupling directions for the acoustically important modes of an instrument. These coupling directions are important factors in determining the efficiency of energy transfer between the vibrating string and the acoustically radiating structural modes of an instrument.

On many stringed instruments several strings are supported by a common bridge and interactions between these strings may be important, as recently discussed by Weinreich [13] for the pairs and triplets of strings forming a single note on the piano. We consider the influence of such interactions on the resonances of a string interacting with a second, but not necessarily identical, string tuned to resonate at the same frequency. We explain how such interactions are utilised by the experienced string player to produce a vibrato effect on the lowest open string of an instrument when bowed.

Finally, after considering a number of factors that may complicate measurements in practice, we argue that the high resolution measurement of string resonances over a wide range of frequencies provides a potentially valuable new technique for the accurate and reliable determination of the mechanical properties of a stringed instrument that ultimately determine its musical quality.

2. Theory

2.1. Lossy string with rigid end-supports

We first consider a lossy but perfectly flexible string of length \( l \), mass \( m \), stretched at tension \( T \) between two rigid end-supports, and excited by a distributed force \( f(x) \exp(j\omega t) \) per unit length. The amplitude \( y \) of transverse string vibration at a point \( x \) along the length of the string satisfies the following equation

\[
\frac{m}{l} \left( \frac{\partial^2 y}{\partial t^2} + \frac{\omega}{Q_s} \frac{\partial y}{\partial t} \right) - T \frac{\partial^2 y}{\partial x^2} = f(x) \exp(j\omega t),
\]

where the second term is included to account for energy loss by direct acoustic radiation from the string and from mechanical hysteresis. In general \( Q_s \), the Q-value of the string resonance, will vary with frequency.

To satisfy the boundary conditions, we may write

\[
y = \left( \sum_n a_n \sin k_n x \right) \exp(j\omega t),
\]

where

\[
k_n = \frac{n \pi}{l}.
\]

The amplitude of the \( n \)-th Fourier component is therefore given by

\[
a_n = \frac{2}{m} \frac{f_n}{\omega_n^2 - \omega^2 + j\omega \frac{Q_s}{Q_s}},
\]

where

\[
f_n = \int_0^l f(x) \sin k_n x \, dx
\]

and \( \omega_n = ck_n \), where \( c = (T/m)^{1/2} \) is the velocity of transverse waves on the string.
For excitation close to a natural harmonic of the string, the resulting string vibrations will be dominated by the Fourier component of the principal mode excited, provided only that \( f_n \) is not too small. We note that, although the amplitude of the \( n \)-th component will depend on the spatial distribution of the exciting force, its frequency dependence will not.

It is convenient to describe the resonant response of the string in terms of the mechanical admittance, which we define as

\[
A_n(\omega, x_0) = [\hat{y}/F_n \exp(i\omega t)]_{x_0},
\]

where \( \hat{y} \) is the velocity of transverse string displacement at the point of application \( x_0 \) of a localised force \( F_n \exp(j\omega t) \).

In practice, a string is usually excited by a spatially distributed force rather than a point force. For example, in refs. [5] to [7] a metal covered string was excited by passing a sinusoidal current through the string, which was placed between the poles of a magnet to produce a Lorentz force over the short length of string in the magnetic field. Even when a distributed exciting force is used to excite the string, it is still possible to describe the response in terms of the admittance defined above, provided we consider an equivalent localised force \( F_n \exp(j\omega t) \) at \( x_0 \) that would have produced the same amplitude of the principal mode excited. \( F_n \) is therefore given by

\[
F_n \sin k_n x_0 = f_n.
\]

For excitation frequencies near the \( n \)-th string resonance the admittance can therefore be written as

\[
A_n = \frac{2}{m} \frac{j\omega}{\omega_n^2 - \omega^2 + j\omega \omega_0^2/Q_s} \sin^2 k_n x_0.
\]

The frequency dependencies of the real and imaginary parts of the above expression are given by the familiar absorption and dispersion curves plotted in Fig. 1, which are normalised to unity at resonance.

In deriving the above result, we have ignored contributions from the weakly excited non-resonant modes of string vibration. For a point driving force an exact expression can be derived for the admittance, either by summing eq. (2.6) over all \( n \)-values [14] or, more straightforwardly, by standard lossy transmission line analysis, which gives

\[
A = \frac{j\pi}{m \omega_1} \frac{1}{(1 - j/Q_s)^{1/2}} \frac{\sin kx \sin k(l-x)}{\sin kl},
\]

(2.7)

where \( k_n \) is calculated from the boundary condition
at the resistive support,
\[-T \left( \frac{\partial y}{\partial x} \right)_I = R \left( \frac{\partial y}{\partial t} \right)_I. \tag{2.9} \]

To first order we therefore obtain
\[b_n l = \frac{Z_0}{R}. \tag{2.10} \]

Once again ignoring contributions from the non-resonant modes and dropping corrections of order \(1/Q^*\), the admittance is given by
\[A_n = \frac{2}{m} \frac{j \omega}{\omega_n - \omega^2 + j \frac{\omega^2}{Q^*} \sin^2 k_n x_0}, \tag{2.11a} \]

where
\[\frac{1}{Q^*} = \frac{1}{Q_n} + \frac{2}{n \pi} \frac{Z_0}{R}. \tag{2.11b} \]

The main effect of the resistive end support is therefore to introduce some additional damping of the resonance.

The corrections to the above result from the non-resonant modes, like the correction terms dropped in deriving eq. (2.11a), are of order \(1/Q^*\), as may be verified by comparison with the exact result for a point force obtained from transmission line theory,
\[A = \frac{2}{m \omega_1 \sqrt{1 - j/Q_n}} \sin k x \left\{ \frac{\sin k(l - x) - j \frac{Z_0}{R} \cos k(l - x)}{\sin k l - j \frac{Z_0}{R} \cos k l} \right\}, \tag{2.12} \]

where \(k\) and \(\omega_1\) are defined for eq. (2.7).

As in the case of the lossy string between rigid end-supports, the correction terms only become important near the nodes of the principal resonance excited, where the admittance curves become increasingly asymmetric. This is shown in Fig. 2, where the admittance curves are derived from the exact transmission line model for excitation of the fundamental \((n = 1)\) at a distance \(l/Q^*\) from the bridge. Provided the admittance is measured at a distance much greater than \((Z_0/R)l\) from a nodal position, the corrections to eq. (2.11a) can usually be ignored. If this is not possible, a detailed mathematical evaluation of the transmission line result may be required, but such a calculation is only strictly applicable for a point driving force. For the remainder of this paper we confine our attention to situations in which we can neglect such corrections of order \(Z_0/R\).

Fig. 2. Asymmetric admittance curves for a string terminated at one end by a resistive support and excited towards a node of the principal resonance excited. The vertical scale is arbitrary.

For strings on a musical instrument the terminating bridge impedance will in general have both resistive and reactive components. Close to a particular structural resonance of the supporting body the effective bridge impedance can be represented by a series resonant circuit, with effective mass \(M\), compliance \(C\), and mechanical resistance \(r\),

Fig. 3. (a) The equivalent circuit for a string, represented by a length of transmission line, coupled at the bridge to a structural resonance, represented by a series resonant circuit.
(b) An equivalent representation of the above situation with the series resonant circuit represented by a short length of resistively terminated transmission line.
as shown in Fig. 3a, where the string is represented by a length of transmission line. Standard transmission line theory allows us to represent the series resonant circuit by a short length of resistively terminated transmission line giving the same impedance as the resonant circuit near resonance, as illustrated schematically in Fig. 3b. The effective length of transmission line $\varepsilon$ (which can have positive or negative values) and its terminating resistance $R$ can, with a little algebra, be shown to be given by

$$k_n \varepsilon = \frac{Z_0}{r} \frac{2 Q_B \delta}{[1 + (2 Q_B \delta)^2]}, \quad (2.13)$$

and

$$R = r[1 + (2 Q_B \delta)^2], \quad (2.14)$$

where $Q_B$ is the $Q$-value of the structural resonance, $\delta \approx (\omega_B - \omega)/\omega_B$ and corrections of order $Z_0/r$ have been ignored.

Close to a structural resonance the coupling therefore causes a slight perturbation of the effective length of the string, and hence its resonant frequency, and produces some additional damping. The frequency response near resonance will be given to a good approximation by eq. (2.11a) with values for the effective length and terminating resistance given by eqs. (2.13) and (2.14).

2.3. Coupling to a structural resonance

Although the influence of the coupled structural resonance can be considered by the transmission line analogue introduced above, it is rather more instructive to consider the dynamics of the mechanical coupling between the string and structural resonances.

We continue to assume that the coupling between the string and the structural resonance is relatively small so that it may be treated as a small perturbation. Close to the $n$-th string resonance we therefore assume that to a good approximation the string vibrations are given by

$$y = a_n \sin(k_n x)e^{i \omega t}, \quad (2.15)$$

with $k_n \approx n \pi/l$.

To first order, the string exerts a force $k_n T a_n \cdot \exp(j \omega t)$ on the bridge causing it to move with a displacement $z \exp(j \omega t)$, where

$$z = \frac{T k_n a}{M (\omega_B^2 - \omega^2 + j \omega^2/Q_B)}, \quad (2.16)$$

and, for simplicity of notation, we consider only odd values of $n$.

To simplify the analysis we assume that the bridge moves in the same direction as the force exerted on it and that only a single structural resonance is involved in the coupling. Both these conditions will be relaxed in subsequent discussion of the problem.

Energy balance considerations enable us to write the following equation for the corresponding amplitude of string vibration,

$$a = \frac{2(F_n \sin k_n x_0 + T k_n z)}{m(\omega_n^2 - \omega^2 + j \omega^2/Q_B)}. \quad (2.17)$$

Eliminating $z$ from eqs. (2.16) and (2.17), we obtain

$$a = \frac{2}{m} \frac{F_n \sin k_n x_0}{\left(\omega_n^2 - \omega^2 + j \omega^2/Q_B\right) - \frac{x^2}{\omega_n^2 - \omega^2 + j \omega^2/Q_B}}, \quad (2.18a)$$

where

$$x^2 = \frac{2(k_n T)^2}{m M} - \frac{\omega_n^4}{(n \pi)^2 M}. \quad (2.18b)$$

The above equations are essentially the same as those obtained by Meamari [15] using a discrete component electrical circuit analysis, apart from the slight generalisation to include intrinsic damping of the string.

Close to a natural string resonance, the admittance is given by

$$A_n = \frac{2}{m} \frac{\omega_n^2}{(\Omega^2_n - \omega^2 + j \omega^2/Q^*)} \sin^2 k_n x_0, \quad (2.19a)$$

where

$$\Omega^2_n = \omega_n^2 - \frac{(\omega_B^2 - \omega^2)x^2}{(\omega_B^2 - \omega^2)^2 + (\omega^2/Q_B)^2}, \quad (2.19b)$$

and

$$\frac{1}{Q^*} = \frac{1}{Q_s} + \frac{x^2/Q_B}{(\omega_B^2 - \omega^2)^2 + (\omega^2/Q_B)^2}. \quad (2.19c)$$

For string resonances close to $\omega_B$, the perturbation of the resonant frequency and the additional damping described by eqs. (2.19b) and (2.19c) are clearly the same as given by eqs. (2.13) and (2.14). The perturbation in resonant frequency is easily understood in terms of the induced motion of the bridge at the point of string support. When the string resonance is at a lower frequency than that of the coupled structural resonance, the bridge moves in the same phase as the force acting on it, so that the effective length of the string is increased ($\varepsilon > 0$), as illustrated schematically in Fig. 4a: its
resonant frequency is therefore decreased. For a string resonance above the resonant frequency of the coupled resonance, the bridge moves in the opposite phase to that of the force acting on it, so that the effective length of the string is decreased ($x < 0$), as indicated in Fig. 4b: its resonant frequency is therefore increased.

From eq. (2.19b) we note that the maximum perturbation of the resonant frequency of the string occurs when $(\omega_B - \omega)/\omega = 1/2Q_B$, where the perturbation is given by

$$\frac{\Delta \Omega_s}{\Omega_s} = \frac{1}{4} \frac{Q_B \omega^2}{\omega_n^2} \approx \frac{1}{2} \frac{Z_0}{\pi r} \frac{\omega}{n} \frac{m}{2 M}.$$  \hspace{1cm} (2.20)

In a subsequent paper [11] we demonstrate that for instruments of the violin family, coupling to the structural resonances can perturb the frequencies of the first few resonances of the lower strings by several cents. The resulting anharmonicity of the first half-dozen or so resonances of a string can be much larger than that caused by the finite flexibility of the string [4], [16], which is often considered the major source of anharmonicity in the resonances of strings on musical instruments.

For relatively weakly coupled string resonances, eq. (2.19c) indicates that the maximum damping occurs when the frequency of the string resonance coincides with that of the structural resonance. If we assume that the intrinsic damping of the string is sufficiently small to be ignored, the effective $Q$ of the string at coincidence is given by

$$Q^* = \frac{\omega^2}{\alpha^2} \frac{Q_B}{2} = \frac{n \pi r}{2} \frac{r}{Z_0} = \frac{(n \pi)^2}{2} \left( \frac{M}{m} \frac{1}{Q_B} \right).$$ \hspace{1cm} (2.21)

Eq. (2.19c) shows that the additional damping will be halved for string resonances shifted a distance $\Delta \omega/\omega = 1/2 Q_B$ from the structural resonance, the frequency where the perturbation in resonant frequency is largest. By measuring the amplitude of string resonances and the associated $Q$-values for a string of known mass in the neighbourhood of a structural resonance, it should be possible to derive values for $\omega_B$, $Q_B$ and $M_B$ for the coupled structural resonance, as indicated schematically in Fig. 5.

In the above analysis, we have inherently assumed that the coupling is relatively weak, so that it is reasonable to describe string resonances by eq. (2.19a-c) with values for the perturbed frequency and additional damping effectively constant over the width of the string resonance. However, for sufficiently strong coupling, this assumption is no longer justified.

We note that the perturbed resonant frequency given by eq. (2.19b) is itself a function of excitation
frequency and is therefore not an eigenfrequency of the coupled oscillator system. For example, if we consider a string resonance initially centred on a strongly coupled structural resonance at $\omega_B$, we note that, as the excitation frequency is moved away from $\omega_B$, the perturbed resonant frequency $\Omega_s$ of the string moves in the same direction, eq. (2.19b), but initially moves away from $\omega_B$ even faster, to frequencies where the damping is decreased, eq. (2.19c). As a consequence, the amplitude of string vibration can increase as the excitation frequency is moved away from $\omega_B$ in either direction, leading to the well-known double resonance of the normal modes of a coupled oscillator system.

3. Normal modes of coupled system

To describe the resonant response in the strongly coupled regime, it is convenient to express the admittance given by eqs. (2.19a-c), in the following equivalent form

$$A_n = \frac{2}{m} C(\omega) \left[ \frac{j \omega}{\Omega_+ - \omega^2 + j \frac{\omega}{\Omega_+ - \omega^2}} \right] \sin^2 k_n x_0,$$

(3.1)

where $\Omega_+$ are the solutions of the equation giving the frequencies of the normal modes of the coupled system,

$$\left( \omega_n^2 - \omega^2 + j \frac{\omega}{\Omega_+} \right) \left( \omega_n^2 - \omega^2 + j \frac{\omega}{\Omega_-} \right) = x^2 = 0,$$

(3.2)

and $C(\omega)$ is a slowly varying function close to unity near the resonant frequencies of the normal modes,

$$C(\omega) = \left[ \frac{\omega_n^2 - \omega^2 + j \frac{\omega}{\Omega_+}}{\Omega_+^2 + \Omega_-^2 - 2 \omega^2} \right].$$

(3.3)

To first order in $1/Q$ and $x^2$, the frequencies of the normal modes are given by

$$\Omega_{\pm}^2 = \omega_n^2 \pm (\omega_n^2 + x^2)^{1/2},$$

(3.4a)

where

$$\omega_n^2 = \frac{\omega_n^2}{2} + \frac{j}{2} \left( \frac{\omega_n^2}{Q_B} \pm \frac{\omega_n^2}{Q_s} \right).$$

(3.4b)

We consider first the solutions of the above equations when the unperturbed string and structural resonances coincide, $\omega_n = \omega_B$. The frequencies of the normal modes are then given by

$$\Omega_{\pm}^2 = \omega_B^2 + j \frac{\omega_B^2}{2 Q_+} \pm \left[ x^2 - \left( \frac{\omega_B^2}{2 Q_-} \right)^{2} \right]^{1/2},$$

(3.5a)

where

$$1 = \frac{1}{Q_B} \pm \frac{1}{Q_s}.$$  

(3.5b)

From the above equations, we see that the character of the normal modes depends not only on the coupling strength $\alpha$ but also on the damping of the coupled structural resonances. The magnitude of $2xQ_\pm/\omega_B^2$ relative to unity determines whether the system can be considered in the strong or weak coupling regime. If we ignore intrinsic damping of the string, the above parameter is essentially the same as the coupling parameter $K$ introduced by Meamari [15]. A further useful identity is $K = (2\Omega_B/n \pi)(2m/M)^{1/2}$.

In the weak coupling limit, $K < 1$, the coupling does not perturb the frequencies of the two normal modes when the unperturbed frequencies of the string and structural resonances coincide. However, the damping of the two modes is modified by the coupling, as expected from our previous discussion.

In the strong coupling limit, $K > 1$, the coupling splits the resonant frequencies of the normal modes symmetrically about the unperturbed resonant frequencies, but the damping of both modes is now the same, with an effective $Q$-value of $2Q_B$, assuming intrinsic string damping can be ignored. In this limit
the normal modes cannot be described as separate string and structural resonances. The low frequency mode involves the simultaneous motion of the string and structural resonances vibrating in the same phase, whereas the high frequency mode involves their motion in anti-phase. These two modes can be represented schematically by Figs. 4 a and b.

In Fig. 6, we have plotted values for the splitting of the frequencies of the normal modes when the unperturbed string and structural resonances coincide. Values are given as a function of harmonic excited, \( n \), and the ratio of the masses, \( m/M \), for a range of \( Q \)-values known to be important when discussing the resonances of instruments of the violin family (see, for example, Firth and Buchanan [17] and [4], [7]). In the extreme strong coupling limit, the modes are split by an amount \( \pm \delta \Omega/\omega_B \) that approaches

\[
\frac{x}{2\omega_B^2} = \frac{1}{n\pi} \sqrt{\frac{m}{2M}} = \frac{K}{4Q_B}.
\]

In the low coupling limit, the shapes of the resonance curves given by eq. (3.1) are almost indistinguishable in practice from the curves for a single resonance illustrated by Fig. 1.

In contrast, Fig. 7 shows a typical example of resonance curves, calculated from eq. (3.2), for the strong coupling limit, which exhibit the double resonance referred to above, with the imaginary component passing through zero at three frequencies symmetrically placed about the unperturbed resonant frequencies. Schellong [4] argued that this multiple zero-crossing feature was the origin of the wolf-note phenomenon. However, recent work has tended to suggest that any appreciable anharmonicity amongst the lowest resonances of a system is likely to lead to instabilities in the oscillations for non-linear excitation processes, such as the bowing of a string or the blowing of a wind instrument (see Benade [2], chapter 25, and Fletcher [18]).

The curves shown in Fig. 7 were evaluated from eq. (3.1) for excitation at the mid-point of the string, where both normal modes (with \( n = 1 \)) are excited with equal amplitudes of vibration. However, in practice, it is often important to consider the admittance nearer the bridge, where, for example, the string would be bowed. Such a position is closer to the effective mode of the high frequency normal mode than it is to the node of the low frequency normal mode. Both the velocity of the string for a given amplitude of excitation and the
rate of energy transferred to the string at the point of excitation (force × velocity) will therefore be smaller for the high frequency mode than for the low frequency mode. The low frequency mode will thus be excited more strongly than the high frequency mode and the admittance will vary as

$$A_n = \frac{2}{m} i \omega \left( \frac{C_+ (\omega) \sin^2 k_\perp A_+}{Q_\perp^2 - \omega^2} + \frac{C_- (\omega) \sin^2 k_\perp A_-}{Q_\perp^2 - \omega^2} \right),$$

(3.6)

where $A_\pm$ are the distances between the point of excitation and measurement and the respective nodes of string motion and $C_\pm (\omega)$ are slowly varying functions of frequency close to unity at the resonances of the normal modes. We continue to assume that the point of excitation is still sufficiently far from either node for the resonances to remain symmetrical ($A_\pm \gg l/Q_\parallel$).

The above predictions are confirmed by curves derived by numerical evaluation of the standard transmission line model. One such set of curves is shown in Fig. 8, evaluated for the same parameters used in deriving Fig. 7, except that the point of measurement $x_0$ is at a distance $l/10$ from the bridge. As a consequence of the weaker excitation of the upper frequency normal mode, the zero-crossing frequencies are no longer symmetrically placed about the unperturbed frequency. For excitation even nearer the bridge, there would be only one zero-crossing point, on the low frequency side of the unperturbed string resonance.

Although it is now widely recognised that a complete explanation of the wolf-note phenomenon requires a rather more sophisticated approach than originally provided by Schelleng [4], the criterion introduced by Schelleng for the occurrence of the wolf-note — that the imaginary component of the reactance (and therefore of the admittance also) should pass through zero three times — appears to be satisfied quite well in practice [4], [7], [15], [17], [19], [20]. By expanding eq. (3.1) about $\omega_0$ in small powers of $\delta \omega$, it is easy to show that, for this criterion to be satisfied, $K \geq 2$, which is twice the value required to produce a splitting of the normal modes.

To extend the above results to string resonances at frequencies away from $\omega_0$, the frequencies of the normal modes must be determined from eqs. (3.4a and b) and inserted in the expression for the admittance, eq. (3.1). In Fig. 9 and Fig. 10 we have plotted curves showing the frequencies of the normal modes as a function of unperturbed string resonant frequency for both a relatively weak coupling situation and a strong coupling situation. The frequency of the normal modes is given by the solid lines and the half-width of their resonances by the dashed curves.
The different character of the solutions in the weak and strong coupling limits is again clearly demonstrated. For coincident unperturbed string and structural resonances the normal modes have the properties described above. For string resonances at higher or lower frequencies, the admittance will tend to be dominated by the response of the normal mode nearest in frequency to the unperturbed string resonance. However, additional structure may also be evident in the wings of the main resonance. In the strong coupling limit such structure persists over an appreciable range about \( \omega_B \), which should therefore enable measurements of string resonances to be used to determine the dispersion of both modes in the vicinity of \( \omega_B \).

4. Polarisation effects

In the previous sections, we made the simplifying assumption that the bridge at the point \( P \) of string support moves in the same direction as the forces produced by the vibrating string. This assumption is not in general justified.

For instruments of the violin family, the most important coupling at low frequencies is to the Helmholtz air resonance and to the main body resonance [2]. These resonances involve a rocking action of the bridge in its own plane about a position close to the foot of the bridge nearest the soundpost. This is illustrated schematically in Fig. 11, where we see that the rocking causes the point \( P \) of string support to move in a particular direction, which we refer to as the rocking direction. Other structural resonances will in general involve a rocking action about different positions and will therefore involve motions of \( P \) in different directions. These directions will be modified by resonances of the bridge itself at higher frequencies, where the bridge can no longer be considered to move as a rigid body (Reiniche [21]).

For the lower string resonances on instruments of the violin family, only those modes of string vibration polarised in the rocking direction associated with the Helmholtz and main body resonances will be appreciably perturbed in frequency or significantly damped by the coupling. Vibrational modes of the string polarised at right angles to this direction will not couple to the principal structural and air resonance and will therefore be only lightly damped and weakly perturbed in frequency by intrinsic string damping and by weak coupling to the higher frequency structural modes. The coupling therefore removes the degeneracy of the transverse vibrations of the string, as confirmed by our earlier measurements [7].

If the string is excited in an arbitrary direction, it will vibrate with components polarised parallel and perpendicular to the principal rocking direction and the admittance curves will exhibit separate resonances for the two components [7]. By adjusting the direction of excitation, it is possible to excite one or other of the two possible modes of string vibration, and therefore to determine the rocking direction associated with the Helmholtz and main body resonances.

In practice, the vibrating string on any musical instrument will be coupled, if only weakly, to a number of structural resonances, each involving a specific motion of the point of string support \( P \). If we specify two orthogonal directions along which the point \( P \) moves with velocities \( u \) and \( v \), making angles \( \theta_\alpha \) and \( (\pi/2) - \theta_\alpha \) with the rocking direction of the \( \alpha \)-th structural resonance, as in Fig. 11, we may write

\[
\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{4.1}
\]

where \( F_1 \) and \( F_2 \) are the forces exerted by the vibrating string at \( P \) in the \( u \) and \( v \)-directions and \( A \) is the tensor

\[
A = \sum_{\alpha} \frac{1}{Z_\alpha} \begin{pmatrix} \cos^2 \theta_\alpha & \cos \theta_\alpha \sin \theta_\alpha \\ \sin \theta_\alpha \cos \theta_\alpha & \sin^2 \theta_\alpha \end{pmatrix},
\]

where

\[
\frac{1}{Z_\alpha} = \frac{j \omega}{M_\alpha (\omega_\alpha^2 - \omega^2 + j \omega^2 / Q_\alpha)}, \quad \tag{4.2}
\]

Fig. 11. A schematic diagram showing how the vibrations of the front of a violin at frequencies near the main body resonance cause the bridge to rock in its own plane about the foot of the bridge nearest the sound post. Such motion is seen to cause the point \( P \) of string support to move in a specific rocking direction.
A transverse wave with string displacements given by
\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} \sin kx e^{iat}
\]
will exert a force
\[
\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = - (-1)^n kT \begin{pmatrix} 1 \\ a \end{pmatrix} e^{iat}
\]
at P. By substitution in the above equations we obtain the following equation for \(a\),
\[
a^2 + 2a \sum \frac{\cos 2\theta_a}{Z_a} - \sum \frac{\sin 2\theta_a}{Z_a} = 0 .
\] (4.3)

There are therefore two possible solutions for \(a\) with their product equal to \(-1\). If the frequency of string vibration is well outside the half-width of any structural resonance, or if the coupling is dominated by a single structural resonance, the solutions for \(a\) will be real corresponding to two possible modes of string vibration linearly polarised in orthogonal directions. Weinreich, in a helpful discussion on this point, has emphasised that in general \(a\) will be complex, so that the two modes of string vibrations will be elliptically polarized. In practice, however, this ellipticity will be small, unless the frequency of string vibration lies close to two or more structural resonances with very different coupling directions.

The frequencies and damping of the two modes of string vibration can be deduced from the effective terminating impedance for the two modes
\[
A_\pm = \frac{1}{Z} \left[ \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 2 \sin^2 \Phi} \right].
\] (4.4a)

where
\[
\frac{1}{Z} = \sum \frac{1}{Z_a},
\] (4.4b)

and
\[
\sin^2 \Phi = Z^2 \sum \frac{\sin^2(\theta_a - \theta_\beta)}{Z_aZ_\beta} .
\] (4.4c)

For string resonances near a particular structural resonance, the above equations will tend to be dominated by a single term. One mode of the resulting string vibration will then be polarised in a direction close to the rocking direction of the structural resonance most strongly excited, with an effective terminating impedance, \(A_+ \approx 1/Z\), determined largely by the principal structural resonance excited. Measurement of the perturbation of resonant frequency and damping of this mode of string vibration will therefore provide information about the most strongly coupled resonance.

The mode of string vibration polarised in the orthogonal direction will be only slightly perturbed by the weaker coupling to other structural resonances, involving rocking motions in directions different from that of the principal structural resonance excited. The perturbation in frequency and damping of this mode of string vibration is given by the effective terminating admittance \(A_- \approx \sin^2 \Phi/2Z\).

Since the rocking directions will not in general coincide with the direction in which a string is forced in practice to vibrate (e.g. when the string is bowed or plucked), the rocking angle \(\theta_\parallel\) relative to the excitation directions is an important parameter in determining the transfer of energy from the vibrating string to the acoustically radiating structural modes.

Furthermore, as a result of the rocking action, the polarisation of the transverse waves excited on a string will in general be changed on reflection at the bridge. String vibrations excited by bowing will be reflected with a component perpendicular to the bowing direction. Subsequent reflection at the bow will modify the perpendicular force between the bow and string, thus modifying the frictional force involved in the original string excitation. Hancock [10] has also drawn attention to the importance of such processes. At present, all theoretical models for the bowed string, including the important recent work of McIntyre and Woodhouse [12] and Schumacher [23], assume that the string motion is parallel to the direction of the exciting force, which is not usually justified in practice.

Until now we have assumed that one end of the string is rigidly supported. However, this condition is easily relaxed. To consider this we examine the resonances of transverse waves with components
\[
\begin{pmatrix} 1 \\ a \end{pmatrix} \sin k(x + \epsilon) e^{iat}
\]
and assume a terminating admittance of the form given in eq. (4.2) at both ends of the string. It is again easy to show that there are two possible solutions for \(a\), corresponding to two possible modes of string vibration polarised in orthogonal directions. The perturbation of the resonant frequencies of these two modes and their damping can again be determined from eqs. (4.3a-c), where the summation is now taken over the coupled modes at both ends of the string. For a string stopped at one end by the player’s finger, the effective terminating impedance
will be largely resistive and will simply result in additional damping of the string resonances.

5. Coupling between strings

On most stringed instruments several strings are supported on a common bridge. Interactions between the strings induced by the motion of the bridge can lead to a number of interesting effects. For example, Weinreich [13] has recently demonstrated that interactions between the pairs and triplets of identical strings that are simultaneously struck by the hammer on a piano affect not only the sound produced by the piano (see also Benade [21]) but also the tuning of the strings themselves.

Since we are only concerned with interactions induced via the motion of the bridge in the rocking direction, we confine our attention to string vibrations polarised in this direction. A full description of the interaction of two strings coupled to a single structural resonance via the induced motion of the bridge would require the solution of three simultaneous equations describing the three normal modes of the system in the coupling direction. However, we simplify the problem by recognising that the string resonances will usually have a much smaller half-width than that of the structural resonance to which they are coupled. We therefore follow Weinreich [13] in assuming that, close to a particular string resonance, the impedance of the bridge, \( Z(\omega) \), can be considered as a slowly varying function of frequency.

The amplitudes of string vibration, \( a_1 \) and \( a_2 \), with resonant frequencies \( \omega_1 \) and \( \omega_2 \) and the amplitude of induced bridge motion, \( z \), satisfy the following equations

\[
a_1 = \frac{2}{m_1} \frac{T_1 k_1 z + F_0 \sin k_1 x_0}{(\omega_1^2 - \omega^2)}, \quad \text{(5.1a)}
\]

\[
a_2 = \frac{2}{m_2} \frac{T_2 k_2 z}{(\omega_2^2 - \omega^2)}, \quad \text{(5.1b)}
\]

and

\[
z = \frac{T_1 k_1 a_1 + T_2 k_2 a_2}{j \omega Z(\omega)}, \quad \text{(5.1c)}
\]

where, for simplicity, we neglect intrinsic string damping, and assume that \( Z(\omega) \) is the same for both strings and that \( a_1, a_2, z \) and \( F_0 \) are collinear.

Solving the above equations we derive the following equation for the admittance at \( x_0 \) on the first string

\[
A = \frac{2}{m_1} D(\omega) \left[ j \frac{\omega}{Q_+ - \omega^2} + j \frac{\omega}{Q_- - \omega^2} \right] \sin^2 k_1 x_0, \quad \text{(5.2)}
\]

where

\[
Q_+ = \frac{\omega_1^2 + \omega_2^2 - \beta_1 - \beta_2}{2} \pm \frac{1}{2}, \quad \text{(5.3a)}
\]

\[
\cdot \left( (\omega^2 - \omega_1^2 + \beta_2 - \beta_1)^2 + 4 \beta_1 \beta_2 \right)^{1/2},
\]

\[
\beta_{1,2} = \frac{2}{m_{1,2}} \frac{(T_{1,2} k_{1,2})^2}{j \omega Z(\omega)} \quad \text{(5.3b)}
\]

and

\[
D(\omega) = \left[ \frac{\omega_1^2 - \omega^2 - \beta_2}{\omega_1^2 + \omega_2^2 - 2 \omega^2 - \beta_1 - \beta_2} \right]. \quad \text{(5.3c)}
\]

When the unperturbed resonant frequencies, \( \omega_1 \) and \( \omega_2 \), of the two strings are coincident, the frequencies of the two normal modes of the system associated with the string resonances are then given very simply by

\[
\Omega_+ = \omega_1^2 - \beta_1 - \beta_2, \quad \text{and} \quad \Omega_- = \omega_2^2. \quad \text{(5.4)}
\]

The \( \Omega_+ \) mode involves the motion of the two strings vibrating in the same phase. Their combined effect is therefore to produce a larger motion of the bridge and hence a larger perturbation in resonant frequency and damping than for a single string alone. The second mode, at an unperturbed frequency \( \Omega_- = \omega_1 = \omega_2 \), involves the vibration of the strings in opposite phases, so that the net force on the bridge is zero. The resonant frequencies of the strings acting together is therefore unchanged.

In Figs. 12a and b we illustrate the modification of the admittance when a second string is adjusted to resonate with the string on which the measurements are being made. Fig. 12a illustrates the situation when the bridge impedance is purely resistive (e.g. in the immediate vicinity of a structural reso-

Fig. 12 (a) Schematic admittance curves (real part only) showing the influence of a second resonating string supported on the common resistive bridge-support. The dashed curve shows the real part of the admittance for a single string acting alone. The solid curve shows the admittance for the interacting strings exhibiting the two resonances of their normal modes.

(b) As above, but for two strings supported on a reactive bridge-support.
nance). The dashed curve represents the real part of the admittance for the single string and the solid curve the real part of the admittance when the second string is tuned to resonance. Fig. 12b illustrates the same effect for a bridge presenting a largely reactive impedance, which, for the situation illustrated, could arise from coupling to a structural resonance at a somewhat higher frequency.

In both cases, the admittance exhibits a very sharp resonance at the frequency of the unperturbed string resonances, which is superimposed on a much broader resonance similar to that obtained for the single string but with an even larger perturbation in resonant frequency and additional damping.

We have already drawn attention to the importance of the interactions between the pairs and triplets of strings on a piano [2], [13]. Such interactions are also important for bowed stringed instruments, and are used by the experienced string player to simulate the effect of vibrato on the lowest open string when bowed. To produce this effect, the player places his finger on the string above the bowed string, at a position corresponding to a note an octave above the bowed note. As the finger is rolled into the position of exact resonance at the octave, the intensity of sound produced at this harmonic increases appreciably, as may easily be demonstrated. Rocking the finger backwards and forwards across the resonant position produces a cyclic variation in the intensity of sound produced, at twice the rocking frequency of the finger. The overall effect is often indistinguishable from the vibrato used on a stopped string, which for stringed instruments is known to involve a considerable amount of amplitude modulation in addition to the anticipated frequency modulation [24].

The above phenomenon can be explained in terms of the coupling between the sympathetically tuned strings, by first noting that coupling to the various structural resonances of the instrument will result in significant perturbations in the natural resonant frequencies of the string. Consequently, the period of the Helmholtz wave-form excited on the string [4] will not in general be harmonically related to the unperturbed string resonance at $\Omega_\ast$ [18]. The bowing action will therefore produce a string motion with a Fourier component slightly shifted from the resonance of the $\Omega_\ast$ mode, with an amplitude determined by the bow velocity and its position on the string. From Figs. 12a and b we note that the admittance outside the region of the very sharp resonance at $\Omega_\ast$ is decreased when the second string is tuned to resonance. The energy transferred to the string at the first harmonic is therefore increased by the presence of the second resonating string, resulting in the increased intensity of the sound produced by the bowed string.

To consider the resonant response of a string interacting with a second string with a different resonant frequency, we have to consider the normal modes of the interacting string system in much the same way that we considered the normal modes of the string and structural resonances earlier. The character of the normal modes depends on the nature of the coupling at the bridge, as may be seen from the dispersion curves given in Figs. 13 to 15, calculated from eq. (5.3a), where we have again represented the frequencies of the normal modes by solid lines and their half-widths by dashed curves.

If the coupling produced by the bridge is purely resistive, the coupling will tend to pull the frequencies of the normal modes together, so that at coincidence of the unperturbed string resonances, $\omega_1 = \omega_2$, the frequencies of the two normal modes are identical but their damping is different. Fig. 13 illustrates the interesting case considered by Weinreich [13] for the interaction of identical strings, where the resistive coupling causes the frequencies of the normal modes to coalesce over a region close to the cross-over position. Outside this region the two modes are equally damped by the resistive bridge impedance but inside this region the damping of the normal modes approaches maximum and minimum (zero) values at coincidence.

![Graph](image)

Fig. 13. The normal modes of two identical strings coupled together by a resistive end-support. The solid lines are the frequencies and the dashed lines the half-widths of the coupled modes, while the dash-dotted line represents the resonant frequencies of the two strings ignoring interactions between them.
For the more general case of interactions between non-identical strings, the coupling still tends to pull the frequencies of the normal modes together, but they only coincide when the unperturbed frequencies of the separate string resonances are equal, as illustrated for the example shown in Fig. 14. As the unperturbed frequencies approach coincidence, the damping of one mode again approaches zero, while the damping of the other mode increases towards a maximum value. Well away from the cross-over region, the two modes can be considered as separate string resonances damped by the resistive terminating impedance of the bridge.

When the coupling at the bridge is largely reactive, the coupling produces a repulsion between the frequencies of the normal modes, as shown in Fig. 15. We have also indicated, by the dash-dotted lines, the perturbed string frequencies ignoring interactions between the strings. The perturbations illustrated in Fig. 15 would arise from interactions with a structural resonance at some what higher frequency. We note that, when the unperturbed frequencies of the two strings coincide, one of the normal modes is again undamped and occurs at the unperturbed frequency of coincidence, while the second normal mode is more strongly damped and its frequency is more strongly perturbed than that of either string acting alone.

In the above discussion we have considered interactions between strings tuned to nearly identical frequencies, whereas the resonant frequencies of most strings on an instrument, including both the playing lengths and any lengths on the non-playing side of the end-supports, will generally be very different. Away from any resonance, the effect of such strings can be taken into account by evaluating their reactive impedance in the coupling direction using standard transmission line theory. The non-resonant strings slightly modify the values of the effective mass for structural resonances at the point P of string support.

In view of the rather unexpected features observed by Weinreich [13] in his measurements of the transient response of the sound produced by the pairs and triplets of strings simultaneously sounded on a piano, it would be interesting to investigate the interactions between such strings by making high resolution measurements of the admittance of individual strings. Using similar techniques to those already developed for measurements on instruments of the violin family, we might reasonably hope to obtain quantitative information about the structural resonances of the soundboard and the detailed mechanics of energy transfer at the bridge, about which there is surprisingly little published information for this important instrument. The results discussed in this section should provide a framework for the interpretation of such measurements.
6. Complicating factors

In our discussion of string resonances we have been making a number of simplifying assumptions which may not be justified in practice. These include the assumption of ideal string properties, the neglect of coupling at the end supports to torsional modes, and the neglect of bridge motion perpendicular to the length of the string.

Any non-uniformity in cross-sectional area or elastic properties can remove the degeneracy of the modes of a string stretched between rigid end supports. The resulting false response of gut strings was a regular problem for string-players before the more-uniform, metal-covered strings, now almost universally used on bowed stringed instruments, became available. There are two ways of dealing with this problem. Firstly, by twisting the string by a small amount in the appropriate direction to bring the normal modes back into coincidence. This was the solution adopted by Vinen [25] in his classic vibrating-wire experiment, in which he used the splitting of the degeneracy of the string modes, induced by the circulation of superfluid helium around a vibrating wire, to demonstrate the quantisation of a macroscopic quantity (the circulation) for the first time. Secondly, the string can be twisted through a large angle, so that one of the modes is completely removed from the area where it might be troublesome. This solution is advocated by Benade (private communication) for dealing with false guitar strings.

Although on many instruments the torsional motion of the string at the end-supports is purposely prevented, by supporting the string at the bridge or nut in a V-shaped groove, there are several important instruments, such as the guitar and viol, where the playing length of the string is determined by a fret. The player applies downward pressure to the string at a position slightly behind the fret. For motion of the string perpendicular to the fret, the string acts as a rigid support and is therefore a node for transverse vibrations polarised in this direction.

However, transverse string vibrations polarised parallel to the fret will tend to produce a rolling motion of the string over the fret, as illustrated in Fig. 16a. The fret is therefore not a node for string vibrations in this direction. For transverse string resonances at a lower frequency than the fundamental torsional resonance, the motion at the fret will be in phase with the force inducing the motion, as shown in Fig. 16a.

We can consider the resulting perturbation of string resonances polarised parallel to the fret by the equivalent transmission line representation of the problem shown in Fig. 16b. We assume that the player's finger provides a perfect node for both transverse and torsional vibrations at a distance s behind the fret. The characteristic impedance for torsional vibrations is measured in the same units as those for transverse vibrations, by defining it in terms of the tangential velocity of the surface of the string induced by a tangential surface force [26].

The effective terminating impedance at the fret for the transverse vibrations of the main string length is therefore given by

\[
Z_{\text{fret}} = -j Z_0 \cot \frac{\omega}{c} \left( \frac{\sin \frac{\omega}{c} s + l}{\sin \frac{\omega}{cT} \sin \frac{\omega}{cT} s} \right),
\]

(6.1)

which for small s may be written as

\[
Z_{\text{fret}} = -j Z_0 \cot \frac{\omega}{c} s \left( \frac{Z_0 c}{Z_0 c + Z_T cT} \right),
\]

(6.2)

The effective node for transverse vibrations polarised parallel to the fret therefore occurs at a distance

\[
\approx \left( \frac{Z_0 c}{Z_0 c + Z_T cT} \right)^{s}
\]

behind the fret. Using typical values for the param-
eters given by Schelleng [26], the factor in the
brackets will generally lie in the range from 0.03 · · · 0.33. The resonant frequency for string vibrations
parallel to the fret will therefore be lower than that
of vibrations polarised perpendicular to the fret.
The dependence of resonant frequency on the posi-
tion of the player's finger explains why players of the
guitar and lute can produce a vibrato effect even
when the finger is slightly behind the fret.

From an experimental viewpoint, any rolling
motion at the end-supports, of the kind discussed
above, can, unless carefully controlled, give rise to
irreproducibilities in measurements of the admittance,
with the end-supports introducing larger perturba-
tions in the string resonances than those arising from
coupling to the structural resonances, which is our principal interest here. We encountered
such problems in our early measurements using
adjustable end-stops, which is why we subsequently
decided to restrict our measurements on the violin
to the natural harmonics of the total string length
between the nut and bridge, which appear to be
almost ideal end-supports. To obtain string reso-
nances over a continuous range of frequencies we
simply adjust the tension.

Finally, we note that both the structural modes
and the flexural resonance of the bridge itself [21],
produce a motion of the point of string support P in
a direction parallel to the string length. The vib-
trating string couples to motion in this direction
through the periodic vibrations in tension of the
string at twice the vibrational frequency, associated
with the change in length of the string for vibrations
of finite amplitude. This coupling provides an addi-
tional mechanism for energy transfer between the
vibrating string and the acoustically radiating
structural modes.

Although such processes may be important in
practice, they can be safely ignored for measure-
ments of string resonances at sufficiently small am-
plitudes. By determining the variation of string
damping as a function of large amplitude vibra-
tion it should be possible to make a quantitative assess-
ment of the importance of such processes for the amplitudes of string vibration excited in practice.

7. Conclusion

We have shown that high resolution measure-
ments of the admittance of strings mounted on
musical instruments should provide a considerable
amount of valuable information about the struc-
tural and other resonances of an instrument that
ultimately determine its tone and quality. Such
information includes the resonant frequency, Q-
value, effective mass and coupling direction of any
structural resonance that is significantly coupled to
the vibrating string through the induced motion of
the bridge. One advantage of using measurements of
the string admittance for deriving this information
is that no external transducers have to be attached
to the body of the instrument, which might other-
wise affect the parameters being measured, a prob-
lem that has been discussed by Hutchins [27] in a
review of other experimental methods for obtaining
such information.

The results that we have presented are applicable
to string resonances on all types of stringed instru-
ments. In a subsequent paper [11] we will present a
number of experimental results to illustrate many
of the features discussed in this paper. These mea-
surements will be interpreted in terms of the results
obtained here and will be used to derive information about the important acoustic reso-
nances of the instruments studied.

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